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Realization of supersymmetric quantum mechanics in inhomogeneous Ising models

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Abstract. Supersymmetric quantum mechanics is well known to provide, together with the so-called shape-invariance condition, an elegant method of solving the eigenvalue problem of some one-dimensional potentials by simple algebraic manipulations. In the present paper, this method is used in statistical physics. We consider the local critical behaviour of inhomogeneous Ising models and determine the complete set of anomalous dimensions from the spectrum of the corresponding transfer matrix in the strip geometry. For smoothly varying perturbations, the eigenvalue problem of the transfer matrix takes the form of a Schrödinger equation, and, furthermore, the corresponding potential exhibits the shape-invariance property for some known extended defects. In these cases, the complete spectrum is derived by the methods of supersymmetric quantum mechanics.

1. Introduction

The concept of supersymmetry first appeared in quantum field theory and later was used in different areas of physics (cf random systems). The essence of the method is more transparently seen in ordinary quantum mechanics as has been known since the work of Witten [1]. Supersymmetric quantum mechanics (SSQM) provides a unified framework for performing the factorization of the Schrödinger equation, following the pioneering works of Dirac and of Schrödinger ([2–5], for a review see [6]). Furthermore, if the potential in the Schrödinger equation has the property of shape invariance [7], the eigenvalues and the corresponding eigenvectors can be obtained by simple algebraic manipulations and it was found that the well known exactly-solved problems (i.e. those problems which can be rewritten as hypergeometric equations after a suitable change of variable) exhibit the shape-invariance property.

In the present paper, we show a possible new field of application for SSQM. In statistical physics, inhomogeneous systems have been extensively studied in the past decade (for a recent review see [8]). An inhomogeneity can be caused basically in two different ways. Geometrical effects due to the surface shape of the system and/or modified couplings or defects may influence the critical behaviour. The simplest inhomogeneity is the semi-infinite system with a free surface. The universal behaviour in a surface layer with a width of the order of the correlation length is described by a set of local (surface) critical exponents which are different from the bulk ones (see [8]). More generally, the existence of a free

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surface may induce a coupling enhancement between nearest-neighbour spins in a region of some extent close to the boundary and a local modification of the critical behaviour can then be expected. One special type of extended defect was introduced by Hilhorst and van Leeuwen [9]. Here the couplings perpendicular to the surface deviate from the bulk one by a power law A/y^{ω}, y being the distance from the free surface. It follows from a relevance–irrelevance criterion [10, 11] that this type of perturbation is marginal for the two-dimensional Ising model at $\omega = 1$. In this case the critical exponents are A-dependent, as obtained from a number of exact calculations [12–18]. The conformal properties of such systems have been investigated using the plane-to-cylinder conformal mapping, under which the system is mapped onto a strip. Provided the perturbation profile is also properly transformed, the gap-exponent relation [19] and the tower-like structure of the spectrum are preserved [20–22]. This is still the case when the defect extends from a line in the bulk [23–27]. It was later shown, in a first-order perturbation calculation, that the gap-exponent relation is valid for any marginal extended perturbation [28]. On the other hand, the geometrical shape of the free boundary may also lead to a modified critical behaviour. These effects are relevant in the critical behaviour at corners or parabola shaped systems (i.e. such that the boundary curve follows a parabolic law) [29–33].

In the present paper, we consider the two-dimensional Ising model with a marginal Hilhorst–van Leeuwen defect, as well as a related hyperbolic type of defect in the corner geometry, and calculate the corresponding local critical exponents. Using conformal methods, the problems are studied in the strip geometry. Here the spectrum of the transfer matrix is calculated exactly using the method of supersymmetric quantum mechanics.

The set-up of the paper is the following. In section 2, we present a short summary of SSQM and of the concept of shape-invariance of the potential partners. In section 3, we show that in the cylinder geometry, when the Hamiltonian limit is considered, the eigenvalue equations in the continuum limit take the form of supersymmetric Schrödinger equations. The Hilhorst–van Leeuwen problem is considered and the complete spectrum of the transfer matrix is calculated by the method of SSQM. The same calculation is performed for the hyperbolic defect in section 4. In section 5, the critical exponents are calculated and a relation between the two problems through conformal invariance is discussed.

2. Supersymmetric quantum mechanics

The work of Witten [1] has focused considerable interest on supersymmetric quantum mechanics (for recent reviews see [34, 35]). Furthermore, by the concept of shape invariance, Gendenshteĭn [7] has obtained a systematic generalization to Dirac's operator method for the 1D harmonic oscillator problem.

Let us consider the Hamiltonian

$$\hat{\mathcal{H}}_- = -\frac{d^2}{d\zeta^2} + \mathcal{V}_-(\zeta) \tag{1}$$

with a vanishing ground-state energy E_0^-. The ground-state wavefunction is then related to the potential as $\mathcal{V}_-(\zeta) = \psi_0''(\zeta)/\psi_0(\zeta)$. In terms of the superpotential

$$\mathcal{W}(\zeta) = -\frac{d}{d\zeta}\ln\psi_0(\zeta) \tag{2}$$

the Hamiltonian $\hat{\mathcal{H}}_-$ is factorized:

$$\hat{\mathcal{H}}_- = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \left(\mathcal{W}^2(\zeta) - \mathcal{W}'(\zeta)\right) = \hat{Q}^+\hat{Q}^- . \tag{3}$$

Here the prime denotes a derivative with respect to ζ and the charge operators are defined by

$$\hat{Q}^+ = -\frac{\mathrm{d}}{\mathrm{d}\zeta} + \mathcal{W}(\zeta) \qquad \hat{Q}^- = \frac{\mathrm{d}}{\mathrm{d}\zeta} + \mathcal{W}(\zeta) . \tag{4}$$

The partner Hamiltonian

$$\hat{\mathcal{H}}_+ = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \mathcal{V}_+(\zeta) = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \left(\mathcal{W}^2(\zeta) + \mathcal{W}'(\zeta)\right) \tag{5}$$

may then be introduced and is also factorized, $\hat{\mathcal{H}}_+ = \hat{Q}^-\hat{Q}^+$, and there exists a one-to-one correspondence between the spectrum of the two partner Hamiltonians as: $E^-_{n+1} = E^+_n$. If the ground-state wavefunctions of $\hat{\mathcal{H}}_\pm$, which are given by (2) as

$$\psi_0^\pm(\zeta) = \exp\left[\pm\int \mathcal{W}(\zeta)\,\mathrm{d}\zeta\right] \tag{6}$$

are non-normalizable, the ground-state energies of both $\hat{\mathcal{H}}_-$ and $\hat{\mathcal{H}}_+$ are non-zero and $E^-_n = E^+_n$. In this case supersymmetry is broken.

In the following we consider unbroken supersymmetry, i.e. $E^-_0 = 0$ and the potential partners which satisfy the shape-invariance property as

$$\mathcal{V}_+(\zeta, a_0) = \mathcal{V}_-(\zeta, a_1) + R(a_1) . \tag{7}$$

Here a_0 is a parameter of the Hamiltonian, a_1 is some function of a_0, and $R(a_1)$ is a function which does not involve the variable ζ. It is then easy to show that the spectrum of $\hat{\mathcal{H}}_-$ and $\hat{\mathcal{H}}_+$ are simply shifted by the amount of $R(a_1)$ and then, by iterating the shape-invariance relation, one builds a hierarchy of Hamiltonians whose spectra are related as mentioned above. Finally, one finds the eigenvalues of $\hat{\mathcal{H}}_-$ as

$$E^-_n(a_0) = \sum_{k=1}^{n} R(a_k) . \tag{8}$$

The corresponding wavefunctions are obtained by applying the charge operators on the ground-state wavefunction:

$$\psi_n(\zeta, a_0) \sim \hat{Q}^+(a_0)\hat{Q}^+(a_1)\ldots\hat{Q}^+(a_{n-1})\psi_0(\zeta, a_n) . \tag{9}$$

The shape-invariant potentials can be found in the literature [36–41]. The factorization technique was, in fact, originally introduced in the context of ordinary differential equations by Darboux [42–44], and the application of the so-called commutation formula to the Schrödinger equation can already be found in [45].

3. Hilhorst–van Leeuwen model

Consider a semi-infinite two-dimensional Ising model with inhomogeneous nearest-neighbour couplings

$$K(\rho,\theta) = K(\infty) - g\mathcal{Z}(\rho,\theta) \tag{10}$$

where $K(\infty)$ is the bulk critical value. The scale covariance requirement for the inhomogeneity leads to a power-law behaviour for the radial part of the shape function [28]:

$$\mathcal{Z}(\rho,\theta) = f(\theta)/\rho^{\omega} \tag{11}$$

and the perturbation amplitude g, then scales under renormalization as $g' = b^{y_t-\omega}g$ where y_t is the bulk thermal exponent. Here, we use the method of conformal invariance. The deviation from the bulk coupling in the original system $t(z) = K(\rho,\theta) - K(\infty)$ transforms, under the conformal mapping $w = w(z) = u + \mathrm{i}v$, according to $t(w) = |w'(z)|^{-y_t}t(z)$ [20]. With the usual plane-to-cylinder logarithmic conformal mapping $w(z) = \frac{L}{\pi}\ln z$, the semi-infinite system is mapped onto an infinitely long strip of width L with free boundary conditions, and the inhomogeneity (10) becomes

$$K(u,v) = K'(\infty) - g\left(\frac{\pi}{L}\right)^{\omega} \exp\left[\frac{\pi u}{L}(\omega - y_t)\right] f\left(\frac{\pi v}{L}\right) \tag{12}$$

where $K'(\infty)$ is the critical coupling in the modified geometry. If we furthermore assume a marginal inhomogeneity, i.e. such that the perturbation amplitude remains unchanged under a rescaling, one has $\omega = y_t$ and it yields a perturbation which is independent of the u-direction along the strip:

$$K(v) = K'(\infty) - g\frac{\pi}{L} f\left(\frac{\pi v}{L}\right). \tag{13}$$

The prototype of smoothly inhomogeneous systems has been introduced by Hilhorst and van Leeuwen [9]. Here, as an illustration, we recover the results previously obtained by Burkhardt and Iglói [20] by more complicated methods. Consider a two-dimensional semi-infinite Ising model on a square lattice. The couplings K_1 parallel to the surface are constant, while the nearest-neighbour couplings $K_2(y)$ perpendicular to the surface assume a power-law deviation from their bulk critical value (figure (1a)):

$$K_2(y) = K_2(\infty) - \frac{g}{y} \tag{14}$$

where y measures the distance from the free surface. This corresponds to a marginal shape function $\mathcal{Z}(\rho,\theta) = (\rho\sin\theta)^{-1}$. This model has been extensively studied in the two-dimensional classical version [9–16] as well as in its quantum counterpart [17, 18, 20–23] (for a review see [8]). Following Burkhardt and Iglói [20], we transform the inhomogeneity by the logarithmic conformal mapping and the inhomogeneity transforms into a sinusoidal form on the strip:

$$K_2(v) = K_2'(\infty) - \frac{\pi}{L}\frac{g}{\sin\left(\frac{\pi v}{L}\right)} \qquad 0 < v < L. \tag{15}$$

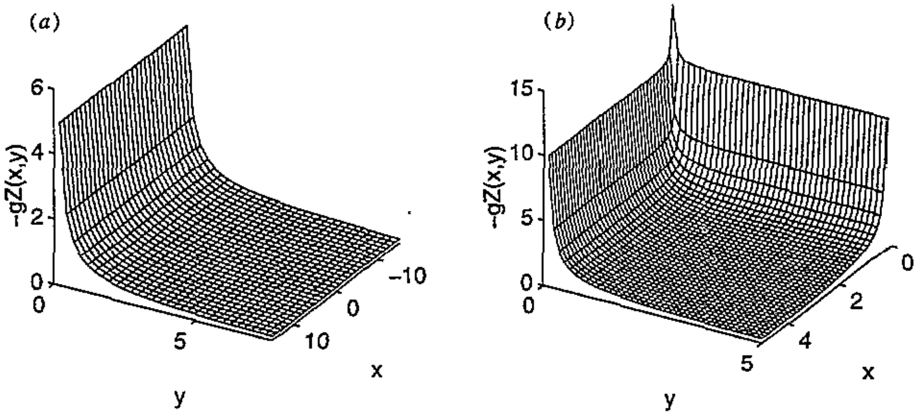

Figure 1. Enhancement of local couplings near an extended defect, (*a*) at a free surface (Hilhorst–van Leeuwen inhomogeneity), (*b*) at a corner (hyperbolic defect).

The transfer matrix along the strip $\hat{\mathcal{T}} = e^{-\tau\hat{\mathcal{H}}}$, where τ is the lattice spacing, leads, in the extreme anisotropic limit [46–48], to a one-dimensional quantum chain defined by the Hamiltonian

$$\hat{\mathcal{H}} = -\tfrac{1}{2}\sum_{\ell=1}^{L}\sigma_z(\ell) - \tfrac{1}{2}\sum_{\ell=1}^{L-1}\lambda(\ell)\sigma_x(\ell)\sigma_x(\ell+1) \tag{16}$$

with varying couplings

$$\lambda(\ell) = 1 - \frac{\pi}{L}\frac{\alpha}{\sin\left(\frac{\pi\ell}{L}\right)}\,. \tag{17}$$

Here, the σ's are the Pauli matrices. The Hamiltonian $\hat{\mathcal{H}}$ can be diagonalized by standard methods [49, 50], transforming in terms of fermion creation (η_k^+) and annihilation (η_k) operators as

$$\hat{\mathcal{H}} = \sum_k \Lambda_k\left(\eta_k^+\eta_k - \tfrac{1}{2}\right). \tag{18}$$

Here, the fermionic modes with the lowest energies, which are $O(L^{-1})$, are obtained in the continuum approximation from a pair of Schrödinger equations involving the inhomogeneity function $\chi(\zeta) = \alpha/\sin\zeta$ where $\zeta = \pi\ell/L$. The first one in terms of ψ_k reads as

$$-\frac{d^2\psi_k}{d\zeta^2} + \left(\chi^2(\zeta) - \chi'(\zeta)\right)\psi_k(\zeta) = \left(\frac{\Lambda_k L}{\pi}\right)^2\psi_k(\zeta) \qquad 0 \leqslant \zeta \leqslant \pi \tag{19a}$$

with the boundary conditions

$$\psi_k(\zeta)|_{\zeta=0} = 0 \qquad \left.\frac{\psi_k'(\zeta)}{\psi_k(\zeta)}\right|_{\zeta=\pi} = -\chi(\pi)\,. \tag{19b}$$

Similarly for the function ϕ_k:

$$-\frac{d^2\phi_k}{d\zeta^2} + \left(\chi^2(\zeta) + \chi'(\zeta)\right)\phi_k(\zeta) = \left(\frac{\Lambda_k L}{\pi}\right)^2\phi_k(\zeta) \qquad 0 \leqslant \zeta \leqslant \pi\,. \tag{20a}$$

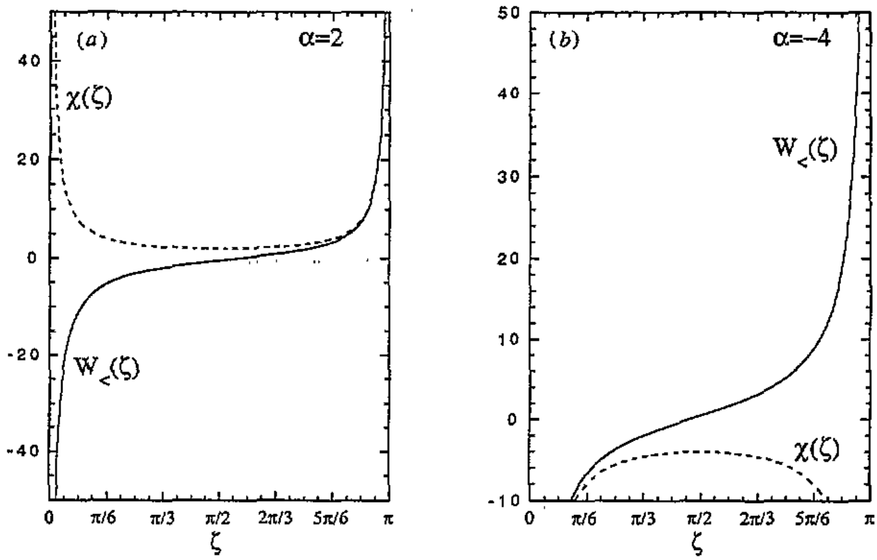

Figure 2. Inhomogeneity function $\chi(\zeta)$ (- - - -) and superpotential $\mathcal{W}(\zeta)$ (——) for the Hilhorst–van Leeuwen model (*a*) for $\alpha = 2$ and (*b*) $\alpha = -4$.

$$\left.\frac{\phi_k'(\zeta)}{\phi_k(\zeta)}\right|_{\zeta=0} = +\chi(0) \qquad \phi_k(\zeta)|_{\zeta=\pi} = 0\,. \tag{20b}$$

In these expressions, $\psi(\zeta)$ and $\phi(\zeta)$ are the continuum limit approximations of the eigenvectors entering the discrete eigenvalue equations that one obtains when diagonalizing the Hamiltonian (16) (see [49]).

The similarity between these equations and the Schrödinger equations encountered in supersymmetric quantum mechanics has already been mentioned by Choi [51], but here we show how the concept of shape invariance may be used to determine the excitation spectrum.

First, we note that all the eigenvalues of (19*a*) and (20*a*) are the same, including the smallest one $(\Lambda_0 L/\pi)^2$, thus, in the language of SSQM, supersymmetry is broken [52, 53]. This statement is in agreement with Witten's argument [1], according to which unbroken supersymmetry requires a superpotential with one node (or an odd number of nodes). This is obviously not the case for $\chi(\zeta)$ (see figure 2), which is symmetrical to $\pi/2$, since the inhomogeneity in the semi-infinite plane is translationally invariant along the surface. We then have to face the problem of finding a superpotential $\mathcal{W}(\zeta)$ in order to restore supersymmetry. This superpotential must be related to the inhomogeneity function $\chi(\zeta)$ by a Riccati equation, i.e. such that $\mathcal{W}^2 - \mathcal{W}'$ and $\chi^2 - \chi'$ are identical up to a constant, the constant being of essential importance because its existence ensures that supersymmetry will be restored.

The boundary conditions in (19*b*) and (20*b*) generally pose the same requirement as in SSQM, i.e. the wavefunction must vanish at both ends of the interval since the potential term diverges there. However, if ψ' diverges faster than ψ when $\zeta \to \pi$, then the solution of (19a) is non-normalizable. This type of solution, which describes a localized mode, is associated with the appearence of spontaneous surface order in the system and corresponds to a vanishing excitation $\Lambda_0 = 0$. For the Hilhorst–van Leeuwen inhomogeneity, such a solution is given by

$$\psi_{\rm loc}(\zeta) \sim \exp\left(-\int \chi(\zeta)\mathrm{d}\zeta\right) = \tan^{-\alpha}\left(\frac{\zeta}{2}\right) \tag{21}$$

which is indeed non-normalizable for $\alpha < -\frac{1}{2}$. The lowest excitation energy in this region is then $\Lambda_0 = 0$. We shall return later to determine the higher-lying levels of the spectrum in this case.

In the following, we deal with the region $\alpha \geqslant -\frac{1}{2}$, where the method of SSQM works without limitations and shape invariance is a worthwhile concept for deducing the eigenvalue spectrum. First, we should find a convenient superpotential which solves the Riccati equation. This is done with the mapping introduced by Dutt *et al* [54]. The inhomogeneity function $\chi(\zeta)$ is a special case of the Scarf superpotential $S(\zeta) = \alpha_1/\sin\zeta - \alpha_2\cot\zeta$, which leads to the Eckart potential by $S^2(\zeta) - S'(\zeta)$ [55]. With the choice $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = \alpha + \frac{1}{2}$, this defines a new superpotential

$$\mathcal{W}_>(\zeta) = -\frac{1}{2\sin\zeta} - \left(\alpha + \tfrac{1}{2}\right)\cot\zeta\,. \tag{22}$$

It is also easy to see that the superpotential $W_>(\zeta)$ presents one node (figure 2), thus Witten's requirement on unbroken supersymmetry is satisfied and supersymmetry is now unbroken in the range $\alpha \geqslant -\frac{1}{2}$. This choice leads to the trigonometric Eckart potential for $\mathcal{V}_-(\zeta)$:

$$\mathcal{V}_-(\zeta) = \frac{\alpha^2 + \alpha\cos\zeta}{\sin^2\zeta} - \left(\alpha + \tfrac{1}{2}\right)^2 \tag{23}$$

and the ground-state excitation Λ_0 can thus be identified as

$$\left(\frac{\Lambda_0 L}{\pi}\right)^2 = \left(\alpha + \tfrac{1}{2}\right)^2. \tag{24}$$

Now equation (19*a*) can be written as

$$-\frac{\mathrm{d}^2\psi_k}{\mathrm{d}\zeta^2} + \left(\mathcal{W}_>^2(\zeta) - \mathcal{W}_>'(\zeta)\right)\psi_k(\zeta) = \left[\left(\frac{\Lambda_k L}{\pi}\right)^2 - \left(\frac{\Lambda_0 L}{\pi}\right)^2\right]\psi_k(\zeta) \tag{25}$$

and the ground-state wavefunction, obtained through (6) is given by

$$\psi_0^>(\zeta) \sim \exp\left(-\int \mathcal{W}_>(\zeta)\,\mathrm{d}\zeta\right) \sim \frac{1}{\sqrt{\pi}}\sin^{\alpha+1}\zeta\ (1+\cos\zeta)^{-1/2}. \tag{26}$$

The solution (26), which is indeed normalizable for $\alpha \geqslant -\frac{1}{2}$, continuously evolves towards the localized mode (21) when $\alpha \to \alpha_c = -\frac{1}{2}$ from above. The higher-lying levels of the Schrödinger equation, which are given as

$$E_k^- = \left[\left(\frac{\Lambda_k L}{\pi}\right)^2 - \left(\frac{\Lambda_0 L}{\pi}\right)^2\right] \tag{27}$$

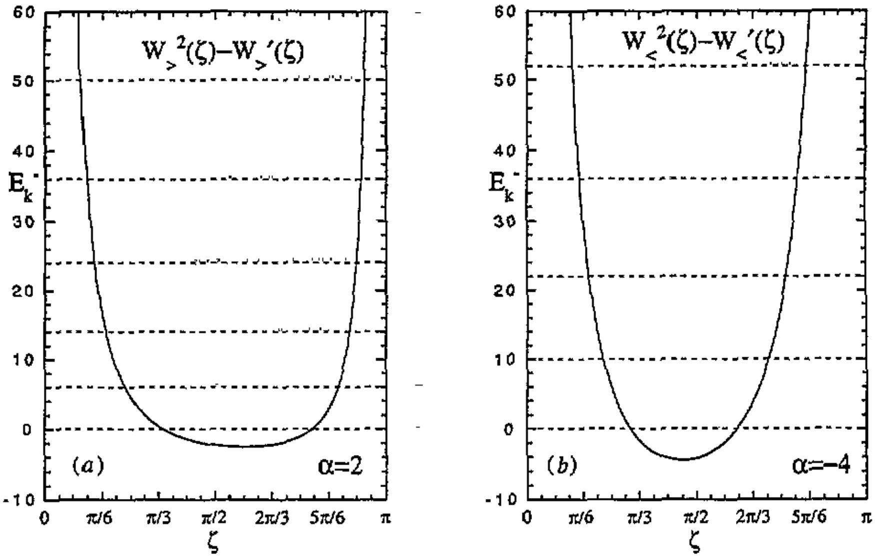

Figure 3. $\mathcal{V}_-(\zeta)$ potential (——) and allowed eigenenergy levels E_k^- (- - - -), (*a*) for $\alpha = 2$ and (*b*) $\alpha = -4$.

are obtained from the shape-invariance property of the partner potentials:

$$\mathcal{V}_+(\zeta, a_0) = \mathcal{V}_-(\zeta, a_1) + \left(a_1 + \tfrac{1}{2}\right)^2 - \left(a_0 + \tfrac{1}{2}\right)^2 \tag{28}$$

where $a_0 = \alpha$, $a_1 = a_0 + 1$. Then, according to (7) and (8), the energies of the single-fermion excitations follow from the remainder function $R(a_1) = (a_1 + \frac{1}{2})^2 - (a_0 + \frac{1}{2})^2$:

$$\Lambda_k = \frac{\pi}{L}\left(\alpha + k + \tfrac{1}{2}\right) \qquad k = 0, 1, 2 \ldots \qquad \alpha \geqslant -\tfrac{1}{2}\,. \tag{29}$$

In the regime of surface order, $\alpha < -\frac{1}{2}$, the eigenfunctions of the excited states of (19*a*) are normalizable, the previous method thus applies. Now, the superpotential is given by

$$\mathcal{W}_<(\zeta) = \frac{1}{2\sin\zeta} + \left(\alpha - \tfrac{1}{2}\right)\cot\zeta \tag{30}$$

and the energy of the first non-vanishing excitation is identified as

$$\left(\frac{\Lambda_1 L}{\pi}\right)^2 = \left(\tfrac{1}{2} - \alpha\right)^2. \tag{31}$$

The higher-lying excitations can be similarly obtained from the shape-invariance property, so that the energies of the fermion modes are now given as

$$\Lambda_0 = 0 \qquad \Lambda_k = \frac{\pi}{L}\left(k - \alpha - \tfrac{1}{2}\right) \qquad k = 1, 2, 3 \ldots \qquad \alpha \leqslant -\tfrac{1}{2}\,. \tag{32}$$

The potential $\mathcal{V}_-(\zeta)$ and the corresponding eigenenergies E_k^- are shown in figure 3 in the ordered phase ($\alpha < \alpha_c$) and in the non-ordered phase ($\alpha > \alpha_c$). Figure 4 shows the two first eigenfunctions in the two regimes.

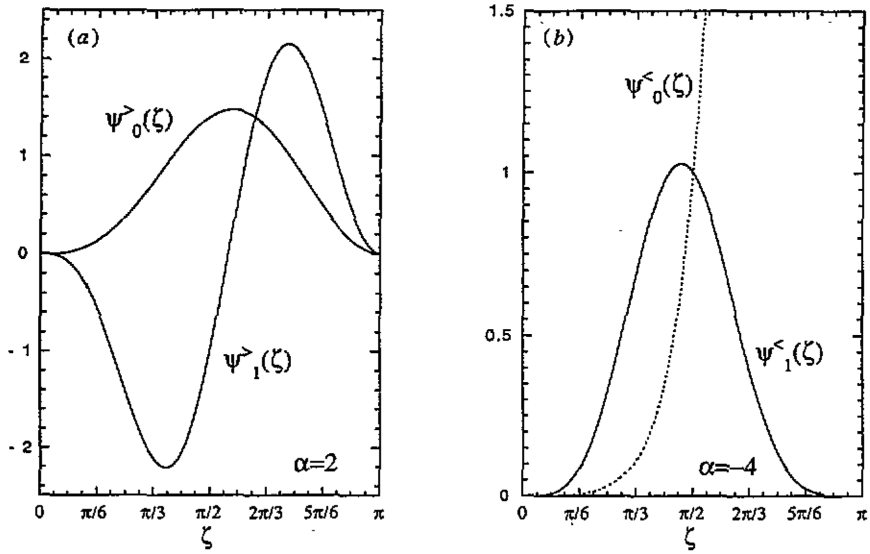

Figure 4. Ground state and first excited wavefunctions for the Hilhorst–van Leeuwen model (a) for $\alpha = 2$ and (b) $\alpha = -4$.

4. Hyperbolic defect

The inhomogeneity in the Hilhorst–van Leeuwen model, studied before, can be considered as a result of elastic deformations on the free surface of the system. If the system now has the shape of a corner with a right angle, then uniform elastic deformations would result in a defect of hyperbolic form. The local couplings are constant along the hyperbolas $f(x, y) = \frac{1}{xy}$ = constant, whereas the couplings pointing perpendicular to the $f(x, y)$ lines are assumed to vary as (figure (1b))

$$K_{\perp}(x, y) = K_{\perp}(\infty) - g\frac{\rho}{xy}. \tag{33}$$

Thus, near to the surface but far from the corner, the inhomogeneity has the same shape as in the Hilhorst–van Leeuwen model in (10). The shape function corresponding to this defect is

$$\mathcal{Z}(\rho, \theta) = \frac{1}{\rho \cos\theta \sin\theta}. \tag{34}$$

Once again, the inhomogeneity is mapped onto a strip geometry, now the appropriate conformal transformation is the $\frac{2L}{\pi}\ln z$ logarithmic mapping. In the strip geometry, the inhomogeneity is again a sinusoidal form, the couplings in the Hamiltonian operator (16) vary as

$$\lambda(\ell) = 1 - \frac{\pi}{2L}\frac{\alpha}{\cos\left(\frac{\pi\ell}{2L}\right)\sin\left(\frac{\pi\ell}{2L}\right)} = 1 - \frac{\pi}{L}\frac{\alpha}{\sin\left(\frac{\pi\ell}{L}\right)} \tag{35}$$

from which we deduce the inhomogeneity function in the continuum limit:

$$\chi(\zeta) = \frac{\alpha}{\cos\zeta \sin\zeta} = \frac{2\alpha}{\sin 2\zeta}. \tag{36}$$

where ζ is defined in the range $0 \leqslant \zeta \leqslant \pi/2$. This inhomogeneity function leads to the Pöschl–Teller potential, but here, from the previous section we can immediately get the energies of the single-particle excitations:

$$\Lambda_k = \frac{\pi}{2L}(2\alpha + 2k + 1) \qquad k = 0, 1, 2 \ldots \qquad \alpha \geqslant -\tfrac{1}{2} \tag{37a}$$

$$\Lambda_0 = 0 \qquad \Lambda_k = \frac{\pi}{2L}(2k - 1 - 2\alpha) \qquad k = 1, 2, 3 \ldots \qquad \alpha \leqslant -\tfrac{1}{2}. \tag{37b}$$

5. Local critical properties

Conformal invariance makes it possible to transform critical systems from one restricted geometry into another, and deduce the local critical exponents in the former geometry from the energy gaps in the transformed one.

In a semi-infinite system, like the Hilhorst–van Leeuwen model, the algebraic decay of the correlation function at the critical point between one point close to the surface ($z_1 \sim 1$) and another point far in the bulk ($z_2 \sim z$) is given asymptotically as $|z|^{-(x+x_1^\mu)}$, where x_1^μ is the surface anomalous dimension of the operator μ, while x is the corresponding exponent for the homogeneous bulk. Under the logarithmic conformal mapping $w(z) = \frac{L}{\pi}\ln z = u + \mathrm{i}v$, the correlation function transforms according to the usual position-dependent law involving only the bulk scaling dimensions:

$$\langle\mu(w_1)\mu(w_2)\rangle = |w'(z_1)|^{-x}|w'(z_2)|^{-x}\langle\mu(z_1)\mu(z_2)\rangle \tag{38}$$

and the correlations in the cylinder geometry exhibit an exponential decay along the strip which defines the correlation length ξ on the strip. In the extreme anisotropic limit, $1/\xi$ is given by the energy gap [47], so that the surface anomalous dimensions are contained in the spectrum of the Hamiltonian operator in (16):

$$x_1^\mu = \frac{L}{\pi}(E_\mu - E_0) \tag{39}$$

as $L \to \infty$. Then, using the diagonal form of $\hat{\mathcal{H}}$ in (18), the surface critical exponents of the Hilhorst–van Leeuwen model can be obtained as combinations of the Λ_k fermion energies. For example the critical exponents of the surface magnetization and surface energy correlations are given by

$$x_1^m = \alpha + \tfrac{1}{2} \qquad x_1^e = 2\alpha + 2 \qquad \alpha \geqslant -\tfrac{1}{2} \tag{40a}$$

$$x_1^m = 0 \qquad x_1^e = \tfrac{1}{2} - \alpha \qquad \alpha \leqslant -\tfrac{1}{2} \tag{40b}$$

in agreement with [20]. Here, $x_1^m = 0$ is due to surface ordering.

For the hyperbolic defect, one defines the corner exponents, denoted x_c^μ, and associated to the algebraic decay of correlations in the corner geometry. In the strip geometry, the x_c^μ's are again proportional to the corresponding gaps of the Hamiltonian operator such as

$$x_c^\mu = \frac{L}{\Theta}(E_\mu - E_0). \tag{41}$$

Then, the corner exponents for the magnetization and the energy for the hyperbolic defect with a right angle $\Theta = \pi/2$ are given as

$$x_c^m = 2\alpha + 1 \qquad x_c^e = 4\alpha + 4 \qquad \alpha \geqslant -\tfrac{1}{2} \tag{42a}$$

$$x_c^m = 0 \qquad x_c^e = 1 - 2\alpha \qquad \alpha \leqslant -\tfrac{1}{2}. \tag{42b}$$

Comparing these results to those of the Hilhorst–van Leeuwen model, one can notice that the corner exponents are, in each case, the double of the corresponding surface ones. The same relation is known between exponents at a free surface and those of a corner of a right angle without the presence of an inhomogeneity, which is, according to Cardy [33], a consequence of conformal invariance. The Schwarz mapping $\tilde{z} = z^{\Theta/\pi}$ with $\Theta = \pi/2$ connects the two geometries and leads to the above relation between the local exponents. It is not difficult to see that the same Schwarz mapping transforms the Hilhorst–van Leeuwen inhomogeneity and the hyperbolic defect into each other, and thus gives the explanation for the observed relation between the corresponding local scaling dimensions. This last result can be used in the opposite direction, then the close relation between the spectrum of the Eckart and that of the Pöschl–Teller potentials can be attributed to conformal symmetry.

Finally we note that the relation between SSQM and inhomogeneous Ising models cannot be exploited further. Inspecting the table of shape-invariant superpotentials [36–41], no further one is known at present which could serve as a basis for a new physically relevant inhomogeneity with an exact solution on the two-dimensional Ising model.

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